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ON THE SPACE–TIME STRUCTURE OF A WAVE FIELD IN A DOUBLE-LAYER MEDIUM WITH AN INTERFACE CRACK†

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Unlike the approaches in [1–3] which treat wave fields over limited times, a technique is developed which enables one, using the principle of a limiting amplitude, to investigate the propagation and establishment of wave fields which are generated in a laminated ideally elastic medium with plane parallel boundaries of separation by the harmonic oscillations of the edges of a crack from an initial instant of time. The need to take account of all the components of the stress tensor outside of the crack in its plane is discussed.

Consider the excitation of wave fields in a bilaminar elastic medium under the action of harmonic stresses from an initial instant of time on the edge of a semi-infinite crack located in the plane of the boundary of separation of the media

$$\mu_j\nabla^2 w_k^j + (\lambda_j+\mu_j)\theta^j_{,k} = \rho_j \partial^2 w_k^j/\partial t^2 \tag{1}$$

$$\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2,\quad \theta^j = \partial w_n^j/\partial x_n$$

$$\mathbf{w}^j = (w_1^j(x_1,x_2,t),\ w_2^j(x_1,x_2,t)) = (u^j, w^j)$$

$$|x_1|<\infty,\ x_2\in[-h_1,h_2],\ \mathbf{x}=(x_1,x_2)=(x,z)$$

$$\Omega = \{x>0,\ z=0\}$$

$$\begin{aligned}
&z=h_2,\ \sigma_{22}^2=0,\ \sigma_{12}^2=0\\
&z=-h_1,\ \sigma_{12}^1=0,\ w_1=0\\
&z=0,\quad w^2-w^1=\begin{cases}0, & \mathbf{x}\notin\Omega\\ \Delta w, & \mathbf{x}\in\Omega\end{cases},\quad u^2-u^1=\begin{cases}0, & \mathbf{x}\notin\Omega\\ \Delta u, & \mathbf{x}\in\Omega\end{cases}\\
&\sigma_{22}^2-\sigma_{22}^1=\begin{cases}0, & \mathbf{x}\notin\Omega\\ \Delta\sigma, & \mathbf{x}\in\Omega\end{cases},\quad \sigma_{12}^2-\sigma_{12}^1=\begin{cases}0, & \mathbf{x}\notin\Omega\\ \Delta\tau, & \mathbf{x}\in\Omega\end{cases}
\end{aligned} \tag{2}$$

$$t=0,\ u^j=w^j=0,\ \partial u^j/\partial t=\partial w^j/\partial t=0 \tag{3}$$

Here x, z is a Cartesian system of coordinates with its origin in the plane of separation of the characteristic properties of the medium, λ_i and μ_i are Lamé parameters, ρ_j are the densities of the lower $(j=1)$ and upper $(j=2)$ layers, $\mathbf{w}^j$, σ_{22}^j and σ_{12}^j are the displacement vector and the components of the stress tensor (normal and shear) in the jth layer, the connection between which is defined by Hooke's law, and Δu, Δw, $\Delta\sigma$ and $\Delta\tau$ are the discontinuities in the

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kinematic and dynamic properties in the plane of discontinuity ($z=0$). It is assumed that the stresses σ_{22}^j and σ_{12}^j are known in the plane of the crack ($\Omega=\{x>0,\ z=0\}$). The formulation of the problem is closed by the condition of the decay of the perturbations at infinity.

When Fourier and Laplace integral transforms are used, the original initial boundary-value problems (1)–(3) reduces, in the case when $\Delta\sigma=0,\ \Delta\tau=0$, to a system of functional equations of the Wiener–Hopf type

$$
\begin{aligned}
&K\Delta W^+ = S^+ + S^- - K_0(T^+ + T^-)\\
&G\Delta U^+ - K\Delta W^+ = G_0(T^+ + T^-),\ \alpha\in E,\ \operatorname{Re} s > s_1 \geqslant 0
\end{aligned}
\tag{4}
$$

where α and s are parameters of the Fourier and Laplace transforms respectively, E is the common strip of regularity of the functions occurring in (4) which contains the whole of the real axis α with the exception, perhaps, of a finite number of points, d is the abscissa of the convergence of the Laplace transform, S^+, T^+, ΔW^+ and ΔS_1 are the Fourier–Laplace images of the functions σ_{22}, σ_{12}, Δw and Δu, respectively, when $\{x>0,\ z=0\}$, and S^- and T^- are the Fourier–Laplace images of the functions when $\{x<0,\ z=0\}$.

Here and henceforth, the plus and minus signs denote the regularity of the functions in the upper ($E\cup\{\alpha:\operatorname{Im}\alpha>0\}$) and lower ($E\cup\{\alpha:\operatorname{Im}\alpha<0\}$) half-planes.

Green's functions, which occur in the system of functional equations (4), are defined by different combinations of the functions Δ_i $(i=1,2,\ldots,5)$

$$
K=\frac{\Delta_1\Delta_2}{\Delta_3},\ K_0=\frac{\Delta_4}{\Delta_3},\ G=\frac{\Delta_1\Delta_2}{\Delta_4},\ G_0=\frac{\Delta_1\Delta_2\Delta_5}{\Delta_3\Delta_4}
\tag{5}
$$

$$
\begin{aligned}
&\Delta_1 = (\alpha^2+\gamma_{1s}^2)^2\operatorname{ch}\gamma_{1p}H_1\operatorname{sh}\gamma_{1s}H_1 - 4\alpha^2\gamma_{1p}\gamma_{1s}\operatorname{sh}\gamma_{1p}H_1\operatorname{ch}\gamma_{1s}H_1\\
&\Delta_2 = 8\alpha^2\gamma_{2p}\gamma_{2s}(\alpha^2+\gamma_{2s}^2)^2[\operatorname{ch}\gamma_{2p}H_2\operatorname{ch}\gamma_{2s}H_2 - 1] -\\
&-[16\alpha^4\gamma_{2p}^2\gamma_{2s}^2 + (\alpha^2+\gamma_{2s}^2)^4]\operatorname{sh}\gamma_{2p}H_2\operatorname{sh}\gamma_{2s}H_2\\
&\Delta_3 = \mu_1\Delta_1T_1 - \mu_2\Delta_2F_1,\ \Delta_4 = i\alpha(\mu_1\Delta_1T_2 - \mu_2\Delta_2F_2)\\
&\Delta_5 = \mu_1^2\Delta_1(T_1T_3 - \alpha^2T_2^2)/\Delta_2 + \mu_2^2\Delta_2F + \mu_1\mu_2(2\alpha^2T_2F_2 - T_1F_3 - T_3F_1)\\
&T_1 = \gamma_{2p}s^2H^2C_{2s}^{-2}((\alpha^2+\gamma_{2s}^2)^2\operatorname{ch}\gamma_{2p}H_2\operatorname{sh}\gamma_{2s}H_2 - 4\alpha^2\gamma_{2p}\gamma_{2s}\operatorname{sh}\gamma_{2p}H_2\operatorname{ch}\gamma_{2s}H_2)\\
&T_2 = [(\alpha^2+\gamma_{2s}^2)^3 + 8\alpha^2\gamma_{2p}^2\gamma_{2s}^2]\operatorname{sh}\gamma_{2p}H_2\operatorname{sh}\gamma_{2s}H_2 -\\
&-2\gamma_{2p}\gamma_{2s}(\alpha^2+\gamma_{2s}^2)(3\alpha^2+\gamma_{2s}^2)[\operatorname{ch}\gamma_{2p}H_2\operatorname{ch}\gamma_{2s}H_2 - 1]\\
&T_3 = \gamma_{2s}s^2H^2C_{2s}^{-2}[(\alpha^2+\gamma_{2s}^2)^2\operatorname{sh}\gamma_{2p}H_2\operatorname{ch}\gamma_{2s}H_2 - 4\alpha^2\gamma_{2p}\gamma_{2s}\operatorname{ch}\gamma_{2p}H_2\operatorname{sh}\gamma_{2s}H_2]\\
&F_1 = \gamma_{1p}s^2H^2C_{1s}^{-2}\operatorname{sh}\gamma_{1p}H_1\operatorname{sh}\gamma_{1s}H_1\\
&F_2 = 2\gamma_{1p}\gamma_{1s}\operatorname{sh}\gamma_{1p}H_1\operatorname{ch}\gamma_{1s}H_1 - (\alpha^2+\gamma_{1s}^2)\operatorname{ch}\gamma_{1p}H_1\operatorname{sh}\gamma_{1s}H_1\\
&F_3 = \gamma_{1s}s^2H^2C_{1s}^{-2}\operatorname{ch}\gamma_{1p}H_1\operatorname{ch}\gamma_{1s}H_1\\
&F = \gamma_{1p}\gamma_{1s}\operatorname{sh}\gamma_{1p}H_1\operatorname{ch}\gamma_{1s}H_1 - \alpha^2\operatorname{ch}\gamma_{1p}H_1\operatorname{sh}\gamma_{1s}H_1\\
&\gamma_{jp}^2 = \alpha^2 + s^2H^2C_{jp}^{-2},\quad C_{jp}^2 = (\lambda_j + 2\mu_j)/\rho_j\\
&\gamma_{js}^2 = \alpha^2 + s^2H^2C_{js}^{-2},\quad C_{js}^2 = \mu_j/\rho_j,\ j=1,2\\
&H = h_1 + h_2,\ H_1 = h_1/H,\ H_2 = h_2/H
\end{aligned}
\tag{6}
$$

By virtue of the uniqueness of the solution of the initial problem, Green's functions (5) are single-valued, analytic functions which are even with respect to the set of variables and do not have any branch points in the complex planes α and s. In this case, their asymptotic behaviour for fixed s and $|\alpha|\to\infty$ is as follows: $K,\ G=O(|\alpha|),\ K_0,\ G_0=O(1)$.

Let us investigate the possibility of factorizing the function K in the form of a product. K is a meromorphic function of the variable α which has an event set of zeros $\alpha=\alpha_m^{\pm}(s)$ and poles $\alpha=\eta_m^{\pm}(s)$ when $s\in D=\{0\leqslant \mathrm{Re}\, s<\epsilon=\mathrm{const},\ -\infty<\mathrm{Im}\, s<+\infty\}$. To be specific, we shall assume that the plus or minus signs for the singularities being considered subsequently denote that they belong to the upper or lower half-planes, respectively.

Let us show that such an approach is possible. Actually, by virtue of the evenness of K with respect to the set of variables, it is possible to fix the branches of the dispersion curves in such a manner that the above-mentioned separation of the dispersion sets is satisfied.

We know that, when $\mathrm{Re}\, s=0$ and $\mathrm{Im}\, s=-\omega<0$, there are a finite number of real values ($\alpha=\eta_m^{+}(-i\omega)$, $m=1, 2, \ldots, M$, $\omega>0$) among the above-mentioned sets and an even set of complex values ($\alpha=\eta_m^{+}(-i\omega)$, $m\geqslant M+1$, $\omega>0$) It follows from the representation

$$\eta_m^+(s) \sim \eta_m^+(-i\omega)+i(\partial\eta_m^+(-i\omega)/\partial\omega)\mathrm{Re}\, s,\quad m=1,2,\ldots,M \tag{7}$$

which holds in the strip D that the condition for fixing $\eta_m^{+}(s)$ in the upper half-plane

$$C_m^{-1}=\partial\eta_m^+(-i\omega)/\partial\omega>0,\ -\infty<\omega<\infty \tag{8}$$

is identical with the natural physical requirement of the positiveness of the group velocity C_m of the corresponding mode. It is obvious that (8) corresponds to the choice of the odd branch of the dispersion curve and, at the same time, that the property, analogous to (7), of η_m^- belonging to the lower half-plane when $m=1, 2, \ldots, M$ follows from the relationship $\eta_m^-(s)=-\eta_m^+(s)$.

A result, similar to that in [4], follows next from the principle of a limit amplitude but is obtained using the limiting-absorptions principle: $\eta_m^{\pm}(s)$, ($s\in D$, $M\geqslant M+1$) do not pass across the real axis and, consequently, are fixed in the upper or lower half-plane, respectively.

The dispersion set of zeros $\alpha=\alpha_m^{\pm}(s)$ also possess the properties considered above.

Hence, on the basis of Weierstrass' theorem, the representation for K, as well as for the new function $L=\Delta_3/\Delta_5$, in the form of infinite products holds, and, consequently, we have their factorization with respect to α in E: $K=K^+K^-$, $L=L^+L^-$, $s\in D$. Then, by using the Wiener–Hopf method, on the basis of the generalized Liouville theorem the images of the functions of the normal and shear stresses outside the crack in its plane can be represented in the form

$$\begin{aligned}S^-(\alpha,s)=\frac{K^-(\alpha,s)}{s+i\omega}\sum_{n=1}^{N}\frac{(\alpha-\alpha_n^+)^{-1}}{[K^-(\alpha_n^+)]'}\{-S^+(\alpha_n^+)+\\+(T^+(\alpha_n^+)+T^-(\alpha_n^-))\Delta_4(\alpha_n^+)/\Delta_3(\alpha_n^+)\}\end{aligned} \tag{9}$$

$$\begin{aligned}T^-(\alpha,s)=-\frac{L^-(\alpha,s)}{s+i\omega}\sum_{m=1}^{M}\frac{(\alpha-\eta_m^+)^{-1}}{[L^-(\eta_m^+)]'}\{T^+(\eta_m^+)+\\+\Delta W^+(\eta_m^+)\Delta_4(\eta_m^+)/\Delta_5(\eta_m^+)\}\end{aligned} \tag{10}$$

Here N is the number of real zeros of the function K, and the coefficients $\Delta W^+(\eta_m^+)$ are defined as the solutions of a system of linear equations (δ_{ml} is the Kronecker delta)

$$A\Delta\mathbf{W}=\mathbf{B},\ \Delta\mathbf{W}=\{\Delta W^+(\eta_m^+)\}_{m=1}^{M} \tag{11}$$

$$\mathbf{A}=\{a_{ml}\}_{m,l=1}^{M},\ \mathbf{B}=\{b_m\}_{m=1}^{M},\quad b_m=b_m^1+b_m^2$$

$$a_{ml}=\delta_{ml}-\frac{\Delta_4(\eta_l^+)/\Delta_5(\eta_l^+)}{K^+(\eta_m^+)[L^+(\eta_l^-)]'}\sum_{k=1}^{M}\frac{\Delta_4(\eta_k^-)/\Delta_5(\eta_k^-)}{K^+(\eta_k^+)[L^+(\eta_k^-)]'}(\eta_m^++\eta_k^+)^{-1}(\eta_k^++\eta_l^+)^{-1}$$

$$b_m^1=\frac{1}{K^+(\eta_m^+)}\sum_{n=1}^{N}\frac{S^+(\alpha_n^+)}{[K^+(\alpha_n^-)]'}(\eta_m^+-\alpha_n^+)^{-1}$$

$$b_m^2 = -\frac{1}{K^+(\eta_m^+)} \sum_{k=1}^{M} \frac{\Delta_4(\eta_k^-)(\eta_m^+ + \eta_k^+)^{-1}}{\Delta_1(\eta_k^-)\Delta_2(\eta_k^-)} \left\{ T^+(\eta_k^-) - L^+(\eta_k^+) \sum_{i=1}^{M} \frac{T^+(\eta_i^+)}{[L^+(\eta_i^-)]'} (\eta_k^+ + \eta_i^+)^{-1} \right\}$$

The construction of the stress functions in the plane of the crack enables one to determine the kinematic and dynamic characteristics of the whole layer. In particular, for the displacements, we derive

$$\begin{aligned}
&\mathrm{w}^j(x, z, t) = \frac{1}{4\pi^2 i} \int_{-\infty}^{\infty} 1^{-i\alpha x} d\alpha \int_{S_0 - i\infty}^{S_0 + i\infty} \mathrm{W}^j(\alpha, z, s) e^{st} ds \\
&\mathrm{W}^2(\alpha, z, s) = H(\mu_2\Delta_2)^{-1} \{ (S^+ + S^-)\mathbf{R}_1(\alpha, z, s) + (T^+ + T^-)\mathbf{R}_2(\alpha, z, s) \} \\
&\mathbf{R}_j(\alpha, z, s) = (R_{j1}(\alpha, z, s),\ R_{j2}(\alpha, z, s)),\ j = 1, 2 \\
&R_{11}(\alpha, H_2, s) = R_{22}(\alpha, H_2, s) = \\
&= -2i\alpha\gamma_{2p}\gamma_{2s}(\alpha^2 + \gamma_{2s}^2)s^2 H^2 C_{2s}^{-2} (\mathrm{ch}\gamma_{2p} H_2 - \mathrm{ch}\gamma_{2s} H_2) \\
&R_{12}(\alpha, H_2, s) = \gamma_{2s} s^2 H^2 C_{2s}^{-2} ((\alpha^2 + \gamma_{2s}^2)^2 \mathrm{sh}\gamma_{2p} H_2 - 4\alpha^2\gamma_{2p}\gamma_{2s}\mathrm{sh}\gamma_{2s} H_2) \\
&R_{21}(\alpha, H_2, s) = \gamma_{2p} s^2 H^2 C_{2s}^{-2} ((\alpha^2 + \gamma_{2s}^2)^2 \mathrm{sh}\gamma_{2s} H_2 - 4\alpha^2\gamma_{2p}\gamma_{2s}\mathrm{sh}\gamma_{2p} H_2)
\end{aligned} \tag{12}$$

For simplicity, let us demonstrate the technique for the inversion of the integrals taking the particular example of a problem with a specified discontinuity in the normal stresses on the crack which neglects the presence of shear stresses in the plane of the crack outside of the crack. In this case, the normal stresses on the boundary of separation outside of the crack can be determined from the first functional equation (4) and the solutions (9)–(11) in Fourier–Laplace images as limiting values when $T^+,\ T^- = 0$

$$\begin{aligned}
&\sigma_{22}(x, t) = \frac{1}{4\pi^2 i} \int_{-\infty}^{\infty} e^{-i\alpha x} d\alpha \int_{s_0 - i\infty}^{s_0 + i\infty} S^-(\alpha, s) e^{st} ds,\quad x < 0,\ z = 0 \\
&S^-(\alpha, s) = -\frac{K^-(\alpha, s)}{s + i\omega} \sum_{n=1}^{N} \frac{S^+(\alpha_n^+)}{[K^-(\alpha_n^+)]'} (\alpha - \alpha_n^+)^{-1}
\end{aligned} \tag{13}$$

The singularities of the integrand in the complex plane s are poles which lie, symmetrically with respect to the origin of coordinates, and on the imaginary axis

$$s = -i\omega, \qquad s = \sigma_m^+(\alpha), \quad \eta_m^+(\sigma_m^+(\alpha)) = \alpha$$

The inner integral is inverted by closure of the integration contour in the left half-plane of the complex parameter s ($\mathrm{Re}\, s < 0$) with subsequent use of Jordan's lemma. We have

$$\begin{aligned}
&\sigma_{22}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S^-(\alpha, t)\, e^{-i\alpha x} d\alpha,\quad x < 0,\ z = 0 \\
&S^-(\alpha, t) = -K^-(\alpha, -i\omega) e^{-i\omega t} \sum_{n=1}^{N} \frac{S^+(\alpha_n^+)}{[K^-(\alpha_n^+, -i\omega)]'} (\alpha - \alpha_n^+)^{-1} - \\
&- \sum_{m=1}^{M} \frac{e^{\sigma_m^+(\alpha) t}}{(\sigma_m^+(\alpha) + i\omega)[K_-^{-1}(\alpha, \sigma_m^+(\alpha))]'} \sum_{n=1}^{N} \frac{S^+(\alpha_n^+)}{[K^-(\alpha_n^+, \sigma_m^+)]'} (\alpha - \alpha_n^+)^{-1}
\end{aligned} \tag{14}$$

It is seen from this representation that the function $S^-(\alpha, t)$ does not have any singularities on the real axis α. At the same time, each of the terms possesses one and the same poles which mutually cancel one another. Hence, in order to pass to term-by-term integration, it is necessary to deform the contour from the real axis into the complex plane close to these singularities. The same kind of deformation can be carried out using the principle of a limiting amplitude, that is, the decay of the solution when $t \to \infty$ which corresponds to the displacement of segments of the initial integration contour close to the negative poles into the upper half-plane α and the initial integration contour close to the positive poles into the lower half-plane. A similar deformation of a contour has been obtained in steady-state harmonic problems on the basis of the limiting amplitude principle [4].

The integral for the first sum in (14), which determines the steady-state contribution to the solution, is evaluated using the theory of residues by closure of the integration contour in the upper half-plane for $x > 0$. Fourier inversion of the second sum after term-by-term integration is carried out using contour integration and subsequent deformation of the contours into the upper or lower half-plane close to the poles of this integrand (14) taking account of the exponential-type decay of the functions in the plane of the space–time parameters x and t. This leads to a cancellation of the contribution from the corresponding mth mode in the domain $|x| > C_m t$, where C_m is the group velocity (8). The solution is finally represented in the form of an expansion with respect to the modes

$$\sigma_{22}(x, t) = e^{-i\omega t} \sum_{m=1}^{M} \frac{e^{i|x|\eta_m^+(-i\omega)}}{[K_+^{-1}(\eta_m^-)]'} \xi(|x|, C_m) \times$$

$$\times \sum_{n=1}^{N} \frac{S^+(\alpha_n^+)}{[K^+(\alpha_n^-)]'} (\eta_m^+ - \alpha_n^+)^{-1} + O(t^{-\nu}) \tag{15}$$

$$x < 0,\ z = 0, \tfrac{1}{3} \leqslant \nu \leqslant \tfrac{1}{2},\ \xi(|x|, C_m) = \tfrac{1}{2}\{\operatorname{sign}(C_m t - |x|) + 1\}$$

We also obtain the remaining representations of the solutions, in a similar manner to (13)–(15), in the form of the sum of the solution of a problem with specified normal stresses in the domain of the crack (Problem 1, index S) and the solution for specified shear stresses in the domain of the discontinuity (Problem 2, index T). Hence

$$w(x, z, t) = w^S(x, z, t) + w^T(x, z, t)$$
$$\sigma_{ij}(x, z, t) = \sigma_{ij}^S(x, z, t) + \sigma_{ij}^T(x, z, t),\quad i, j = 1, 2 \tag{16}$$

In particular, in the case of Problem 1, we have for the displacement field of the surface of the medium and the stresses outside the crack in its plane

$$\mathbf{w}^S = \mathbf{w}^{SS} + w^{ST},\quad z = h_2 \tag{17}$$

$$x > 0$$

$$\mathbf{w}^{SS}(x, h_2, t) = \frac{H}{\mu_2} e^{-i\omega t} \sum_{k=1}^{N^2} \frac{e^{ix\alpha_{k2}^+(-i\omega)}}{\Delta_2'(\alpha_{k2}^-)} \xi(x, v_{k2}) \times$$

$$\times \mathbf{R}_1(\alpha_{k2}^-, H_2, -i\omega)\{S^+(\alpha_{k2}^-) - K^+(\alpha_{k2}^+) \sum_{n=1}^{N} \frac{S^+(\alpha_n^+)}{[K^+(\alpha_n^-)]'} (\alpha_{k2}^+ + \alpha_n^+)^{-1}\}$$

$$\mathbf{w}^{ST}(x, h_2, t) = \frac{H}{\mu_2} e^{-i\omega t} \sum_{k=1}^{N2} \frac{e^{ix\alpha_{k2}^+(-i\omega)}}{\Delta_2'(\alpha_{k2}^-)} \xi(x, v_{k2}) \times$$

$$\times \{-\mathbf{R}_1(\alpha_{k2}^-, H_2, -i\omega) Q(\alpha_{k2}^-) + \mathbf{R}_2(\alpha_{k2}^-, H_2, -i\omega) T^-(\alpha_{k2}^-)\}$$

$$x < 0$$

$$\mathbf{w}^{SS}(x, h_2, t) = \frac{H}{\mu_2} e^{-i\omega t} \sum_{m=1}^{M} \frac{e^{i|x|\eta_m^+(-i\omega)} \mathbf{R}_1(\eta_m^+, H_2, -i\omega)}{[K_+^{-1}(\eta_m^-)]' \Delta_2(\eta_m^+)} \xi(|x|, C_m) \times$$

$$\times \sum_{n=1}^{N} \frac{S^+(\alpha_n^+)}{[K^+(\alpha_n^-)]'} (\eta_m^+ - \alpha_n^+)^{-1}$$

$$\mathbf{w}^{ST}(x, h_2, t) = -\frac{H}{\mu_2} e^{-i\omega t} [\sum_{k=1}^{N2} \frac{e^{i|x|\alpha_{k2}^+(-i\omega)}}{\Delta_2'(\alpha_{k2}^+)} T^-(\alpha_{k2}^+) \xi(|x|, v_{k2}) \times$$

$$\times \{\mathbf{R}_1(\alpha_{k2}^+, H_2, -i\omega)\Delta_4(\alpha_{k2}^+)/\Delta_3(\alpha_{k2}^+) + \mathbf{R}_2(\alpha_{k2}^+, H_2, -i\omega)\} -$$

$$- \sum_{m=1}^{I} \frac{e^{i|x|\zeta_m^+(-i\omega)} \mathbf{R}_2(\zeta_m^+, H_2, -i\omega)}{[L_+^{-1}(\zeta_m^-)]' \Delta_2(\zeta_m^+)} \xi(|x|, d_m) P(\zeta_m^+) +$$

$$+ \sum_{m=1}^{M} \frac{e^{i|x|\eta_m^+(-i\omega)} \mathbf{R}_1(\eta_m^+, H_2, -i\omega)}{[K_+^{-1}(\eta_m^-)]' \Delta_2(\eta_m^+)} \xi(|x|, C_m) Q(\eta_m^+)] \tag{18}$$

$$\sigma_{22}^S = \sigma_{22}^{SS} + \sigma_{22}^{ST}, \quad x < 0, \ z = 0$$

$$\sigma_{22}^{SS}(x, t) = e^{-i\omega t} \sum_{m=1}^{M} \frac{e^{i|x|\eta_m^+(-i\omega)}}{[K_+^{-1}(\eta_m^-)]'} \xi(|x|, C_m) \sum_{n=1}^{N} \frac{S^+(\alpha_n^+)}{[K^+(\alpha_n^-)]'} (\eta_m^+ - \alpha_n^+)^{-1}$$

$$\sigma_{22}^{ST}(x, t) = e^{-i\omega t} \sum_{m=1}^{M} \frac{e^{i|x|\eta_m^+(-i\omega)}}{[K_+^{-1}(\eta_m^-)]'} \xi(|x|, C_m) Q(\eta_m^+)$$

$$\sigma_{12}^{ST}(x, t) = e^{-i\omega t} \sum_{k=1}^{I} \frac{e^{i|x|\zeta_k^+(-i\omega)}}{[L_+^{-1}(\zeta_k^-)]'} \xi(|x|, d_k) P(\zeta_k^+), \quad T^-(\alpha) = L^+(-\alpha) P(\alpha)$$

$$P(\alpha) = \sum_{m=1}^{M} \frac{\Delta W^+(\eta_m^+)\Delta_4(\eta_m^+)/\Delta_5(\eta_m^+)}{[L^+(\eta_m^-)]'(\alpha - \eta_m^+)}, \qquad Q(\alpha) = \sum_{n=1}^{N} \frac{T^-(\alpha_n^+)\Delta_4(\alpha_n^+)/\Delta_3(\alpha_n^+)}{[K^+(\alpha_n^-)]'(\alpha - \alpha_n^+)}$$

We note that the function σ_{22}^{ss} is that defined in (15) as $\sigma_{22}(x, t)$.

Analogous relationships are obtained in the case of Problem 2.

The singularities $\alpha = \zeta_m^{\pm}(-i\omega)$ correspond to the set of zeros of the function Δ_5 defined in (6). The set of singularities $\alpha = \alpha_n^{\pm}(-i\omega)$ are represented in the form $\{\alpha_n^{\pm}(-i\omega)\} = \{\alpha_{n1}^{\pm}(-i\omega)\} \cup \{\alpha_{n2}^{\pm}(-i\omega)\}$, where $\alpha_{n1}^{\pm}$ and $\alpha_{n2}^{\pm}$ correspond to zeros of the functions Δ_1 and Δ_2. Here I, $N1$ and $N2$ ($N = N1 + N2$) are the number of real values of ζ_m^+, $\alpha_{n1}^{\pm}$ and $\alpha_{n2}^{\pm}$, respectively, at a fixed $\omega > 0$ and the group velocities of the corresponding modes are introduced by analogy with (8) using the formulae $d_m^{-1} = \partial\zeta_m^+(-i\omega)/\partial\omega$, $v_{mk}^{-1} = \partial\alpha_{mk}^+(-i\omega)/\partial\omega$, $k = 1, 2$.

The coefficients $\Delta W^+(\eta_m^+)$ are determined from system (11) and, moreover, only the first term b_{m1} is taken as the right-hand side in the case of Problem 1. In the case of Problem 2, the analogous coefficients are determined from (11) with the right-hand side b_{m2}.

In the solution of Problem 1, the first terms with the index SS correspond to the solution in the special case which does not take account of the shear stresses beyond the discontinuity on its continuation while, in Problem 2, the terms with the index TT are the solution in the special

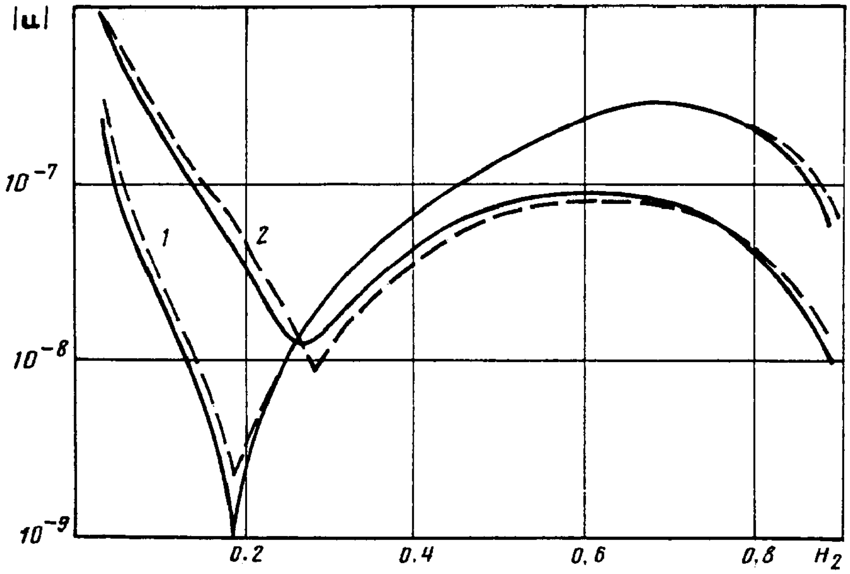

Fig. 1.

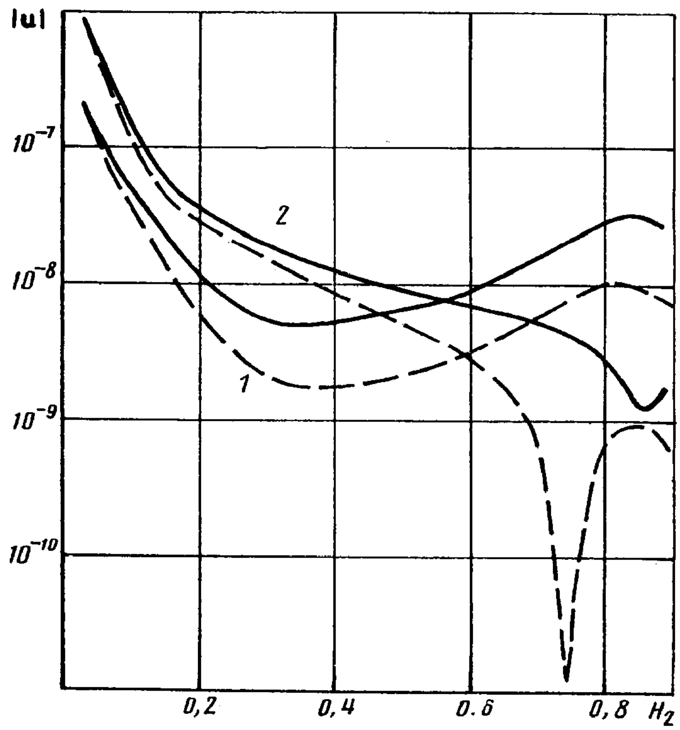

Fig. 2.

case when no account is taken of the normal stresses on the continuation of the discontinuity.

Results of calculations of the amplitudes of the wave fields as a function of the deepening of the crack in Problem 1 allowing for the shear stresses on the continuation of the fracture line (the solid line) and without taking account of them (the dashed line) are presented in Figs 1–3.

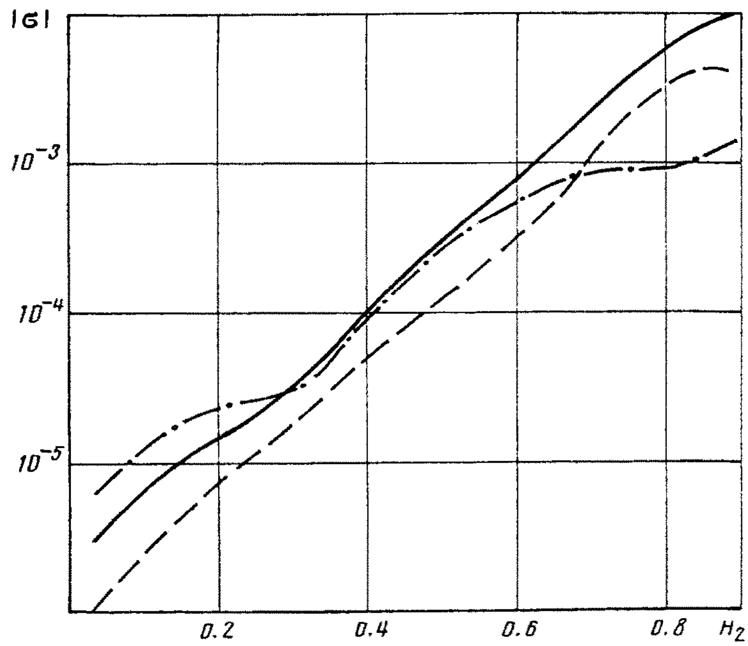

FIG. 3.

The groups of curves 1 and 2 correspond to the vertical or tangential components of the displacements of the free surface above the fracture (Fig. 1) and outside of it (Fig. 2). Graphs of the amplitude functions of the normal stresses are shown in Fig. 3, where the dot–dash line corresponds to the shear stress function on the continuation of the fracture.

The perturbation function is chosen in such a way that its Fourier image does not have any singularities on the real axis in the plane of the complex parameter α.

The following values of the parameters (in dimensionless form) were used in the calculations

$$\omega = 0.57;\ x = 1.33;\ t = 2.33;\ C_{1p}/C_{1s} = 2{,}8571$$
$$C_{1p}/C_{2p} = 1.6;\ C_{1p}/C_{2s} = 5.0;\ \eta = 117 + i20$$

The numerical analysis of this and other versions shows that failure to take account of shear stresses on the continuation of the fracture line in Problem 1 and the normal stresses in Problem 2 does not lead to any appreciable changes in the displacement field above the fracture but, outside of this domain, it can have an appreciable influence on the displacements of the surface and on the stresses on the continuation of the fracture.

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